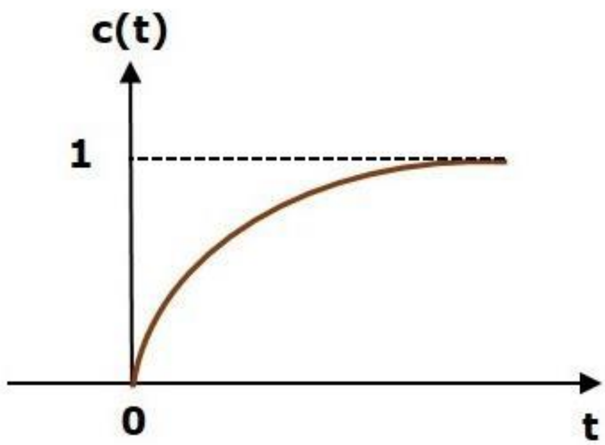
Lecture9: Stability Analysis and Routh's Stability Criterion

9.1 Introduction:

Stability is an important concept. A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable.

A **stable system** produces a bounded output for a given bounded input. The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of **t** including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

9.2 Types of Systems based on Stability:

We can classify the systems based on stability as follows:

9.2.1 Absolutely stable system:

If the system is stable for all the range of system component values, then it is known as the absolutely stable system. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of ‘s’ plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the ‘s’ plane.

9.2.2 Conditionally stable system:

If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

9.2.3 Marginally stable system:

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

9.3 Routh's Stability Criterion:

Routh's stability criterion can be readily applied to the characteristic equation to find out the presence of the roots without having to solve the actual equation (without having to factor the denominator polynomial) i.e Routh's stability criterion enables us to determine the number of closed-loop poles that lie in the right-half s plane.

The procedure in Routh's stability criterion is as follows:

1. Write the characteristic's equation in the following form:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

where the coefficients are real quantities. We assume that an # 0; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable. Note that all the coefficients must be positive. It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. **The necessary but not sufficient condition for stability is that the coefficients of characteristics equation all are present and all have a positive sign.**

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \dots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \dots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

Where n=7 is formed as given below:

$$
\left|
\begin{array}{llll}
a_0 & a_2 & a_4 & a_6 \\
a_1 & a_3 & a_5 & a_7 \\
b_1 & b_3 & b_5 & \\
c_1 & c_3 & & \\
d_1 & d_3 & & \\
e_1 & & & \\
f_1 & & &
\end{array}
\right|
$$

where

$$b_1 = \frac{\begin{matrix} a_0 & a_2 \\ a_1 & a_3 \end{matrix}}{a_1} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_3 = \frac{\begin{matrix} a_0 & a_4 \\ a_1 & a_5 \end{matrix}}{a_1} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_5 = \frac{\begin{matrix} a_0 & a_6 \\ a_1 & a_7 \end{matrix}}{a_1} = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{\begin{matrix} a_1 & a_3 \\ b_1 & b_3 \end{matrix}}{b_1} = \frac{b_1 a_3 - a_1 b_3}{b_1}$$

$$c_3 = \frac{\begin{matrix} a_1 & a_5 \\ b_1 & b_5 \end{matrix}}{a_1} = \frac{b_1 a_5 - a_1 a_5}{b_1}$$

$$d_1 = \frac{\begin{matrix} b_1 & b_3 \\ c_1 & c_3 \end{matrix}}{c_1} = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$d_3 = \frac{\begin{matrix} b_1 & b_5 \\ c_1 & 0 \end{matrix}}{c_1} = \frac{c_1 b_5 - b_1 0}{c_1} = b_5$$

If we study the array successive rows have one term fewer than the preceding row, and hence the array is triangular.

The following are the limitations of the Routh's stability criterion.

- It is valid only if the characteristic equation is algebraic.
- If any coefficient of the characteristic equation is complex or contain power of 'c', this criterion cannot be applied.
- It gives us an information as to how many roots are lying in the right-hand side of the s-plane. Values of the roots is not available. Also, it cannot distinguish between real and complex roots.

Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Note: The necessary and sufficient condition for system to be stable is “all the terms in the first column of Routh’s array must have same sign. There should not be any sign change in the first column of Routh’s array”.

If there are any sign changes existing then:

A. System is unstable.

B. The number of sign changes equals the number of roots lying in the right half of s-plane.

The number of roots of characteristics equation with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Example 1: Using Routh's stability criterion, check whether system represented by the following characteristic equation is stable or not. Comment on the location of roots.

$$s^3 + 20s^2 + 9s + 100 = 0$$

Solution:

$$\begin{array}{c|ll|} s^3 & 1 & 9 \\ s^2 & 20 & 100 \\ s^1 & 4 & \\ s^0 & 100 & \end{array}$$

Since there is no sign change in the first column of Routh's array, no roots of characteristic equations are located on the RHS of s-plane. System is stable.

Example 2: Using Routh's stability criterion, check whether system represented by the following characteristic equation is stable or not. Comment on the location of roots.

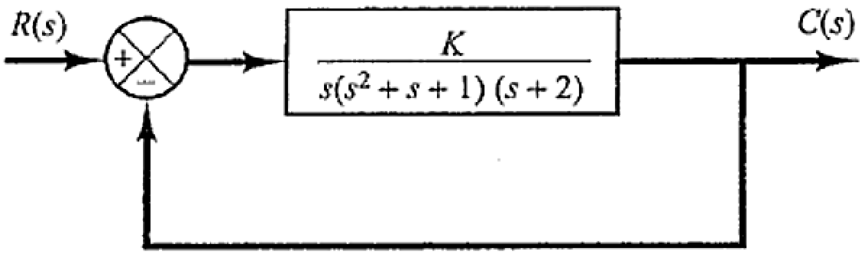
$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Solution:

$$\begin{array}{c|lll|} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array}$$

Since there are two sign changes in the first column of Routh's array, two roots of characteristic equations with positive real parts are located on the RHS of s-plane. System is unstable.

Example 3: Consider the system shown in figure below. Determine the range of K for stability.



Solution: The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

The characteristic equation is:

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes:

$$\begin{array}{c|ccc|}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
s^2 & \frac{7}{3} & K & \\
s^1 & 2-\frac{9}{7}K & & \\
s^0 & K & &
\end{array}$$

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

Note: When K = 14/9, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude. The marginal value of K is a value which makes any row other than s^0 as row of zeros. And to obtain the frequency of oscillation, solve the auxiliary equation P(s)=0 for $K=K_{mar}$.

Note that the ranges of design parameters that lead to stability may be determined by use of Routh's stability criterion.

Note: Auxiliary equation is always the part of the original characteristic equation. This means the roots of the auxiliary equation are some of the roots of original characteristic equation. Not only this but the roots of auxiliary equation are the most dominant roots of the original characteristic equation, from the stability point of view.

Example 4: Consider the following characteristic equation:

$$\boldsymbol{s^4 + Ks^3 + s^2 + s + 1 = 0}$$

Determine the range of K for stability.

Solution: The Routh array of coefficients is:

$$\begin{array}{c|ccc|}
s^4 & 1 & 1 & 1 \\
s^3 & K & 1 & 0 \\
s^2 & \dfrac{K-1}{K} & 1 & 0 \\
s^1 & 1-\dfrac{K^2}{K-1} & 0 & \\
s^0 & 1 & &
\end{array}$$

For stability, we require that:

$K > 0$

$$\frac{K-1}{K} > 0$$

$$1 - \frac{K^2}{K-1} > 0$$

From the first and second conditions, K must be greater than 1. For K>1, notice that the term $1 - \frac{K^2}{K-1}$ is always negative, since:

$$\frac{K-1-K^2}{K-1} = \frac{-1+K(1-K)}{K-1} < 0$$

Thus, the three conditions cannot be fulfilled simultaneously. Therefore, there is no value of K that allows stability of the system.

9.4 Special Cases:

Case1: If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number K and the rest of the array is evaluated. For example, consider the following characteristic's equation:

$$\boldsymbol{s^3 + 2s^2 + s + 2 = 0}$$

The array of coefficients is:

$$\begin{array}{c|ll|} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx K & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (K) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, the above characteristic's equation has two roots at s = + j and s=-j.

If, however, the sign of the coefficient above the zero (K) is opposite that below it, it indicates that there is one sign change.

Example 5: For the characteristic's equation: $\boldsymbol{s^3 - 3s + 2 = (s-1)^2(s+2) = 0}$

Determine the number of changes in sign of the coefficients in the first column of Routh's array.

Solution: The array of coefficients is:

One sign change:

One sign change:

$$\begin{array}{c|ll|} s^3 & 1 & -3 \\ s^2 & 0 \approx K & 2 \\ s^1 & -3 - \dfrac{2}{K} & \\ s^0 & 2 & \end{array}$$

There are two sign changes of the coefficients in the first column.

Example 6: For the characteristic's equation:

$$\boldsymbol{s^4 + 4s^3 + 4s^2 + 3s + K = 0}$$

Determine the value of K that will cause sustained oscillations in the system. Also find the oscillation frequency.

Solution: The Routh's array is:

$$\begin{array}{c|ccc|}
s^4 & 1 & 4 & K \\
s^3 & 4 & 3 & 0 \\
s^2 & \frac{13}{4} & K & \\
s^1 & \dfrac{(\frac{39}{4} - 4K)}{\frac{13}{4}} & & \\
s^0 & K & &
\end{array}$$

The condition for system stability is:

$K > 0$

$$\frac{(\frac{39}{4} - 4K)}{\frac{13}{4}} > 0$$

Therefore, for stability, K should lie in the range:

$$\frac{39}{16} > K > 0$$

When $K = \frac{39}{16}$, there will be a zero at the first entry in the fourth row. This will indicate presence of symmetrical roots, which as shown below, will be pure imaginary.

$K = \frac{39}{16}$ will cause sustained oscillations.

The subsidiary equation of third row for

$K = \frac{39}{16}$, is:

$$\frac{13}{4}s^2 + \frac{39}{16} = 0$$

$\therefore s = \pm j0.75$

Thus, the frequency of sustained oscillations is 0.75 rad/sec.

Example 7: Determine the values of K and b, so that the system whose open loop transfer function is:

$$\boldsymbol{G(s) = \frac{K(s+1)}{s^3 + bs^2 + 3s + 1}}$$

Oscillates at a frequency of oscillations of 2 rad/sec. Assume unity feedback.

Solution: The characteristic equation is:

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s^3 + bs^2 + 3s + 1} = s^3 + bs^2 + 3s + 1 + K(s+1) = 0$$

Or $s^3 + bs^2 + (K+3)s + (K+1) = 0$

The Routh's array is:

$$\begin{array}{c|ll|}
s^3 & 1 & K+1 \\
s^2 & b & K+1 \\
s^1 & (K+3) - \dfrac{(K+1)}{b} & \\
s^0 & K+1 &
\end{array}$$

The system will have a sustained oscillation if row No. 3 is zero i.e.

$$\frac{b(K+3) - (K+1)}{b} = 0$$

i.e. $b = \frac{K+1}{K+3}$ ... (1)

The roots of auxiliary equation at marginal value of K are:

The subsidiary equation of row No. 2 is:

$bs^2 + (K+1) = 0$

$\therefore s^2 = -\frac{(K+1)}{b} = (jw)^2 = (j2)^2 = -4$ (because ω=2 rad/sec)

Putting $b = \frac{K+1}{4}$ in equation (1), we get:

$$\frac{K+1}{4} = \frac{K+1}{K+3} \Longrightarrow K + 3 = 4$$

$K = 1$

$$b = \frac{K+1}{4} = \frac{1+1}{4} = 0.5$$

Case2: If all the coefficients in any derived row are zero, replace a row of zeros by the coefficient of the derivative of this polynomial, then complete the Routh's array.

- If sign change occurs in the first column, system is unstable with number of sign changes equal to number of roots of characteristic equation located in right half of s-plane.
- If there is no sign change, system can not be predicted as stable. And in such case stability is to be determined by actually solving P(s)=0 for its root because roots P(s)=0 are always dominant roots of characteristic equation.

Example 8: Using Routh's stability criterion, check whether system represented by the following characteristic equation is stable or not. Comment on the location of roots.

$$\boldsymbol{s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0}$$

Solution: The array of coefficients is:

$$\begin{array}{c|ccc|} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array}$$

Auxiliary Polynomial P(s)

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial P(s) is:

$$\boldsymbol{P(s) = 2s^4 + 48s^2 - 50}$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex conjugate roots on the imaginary axis). These pairs are obtained by solving the auxiliary polynomial equation **P(s) = 0**. The derivative of **P(s)** with respect to s is:

$$\boldsymbol{\frac{dP(s)}{ds} = 8s^3 + 96s}$$

The terms in the s^3 row are replaced by the coefficients of the last equation, that is, **8** and **96**. The array of coefficients then becomes:

$$\begin{array}{c} s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{array}$$

Coefficients of dP(s)/ds

$$\begin{vmatrix} 1 & 24 & -25 \\ 2 & 48 & -50 \\ 8 & 96 & \\ 24 & -50 & \\ 112.7 & 0 & \\ -50 & & \end{vmatrix}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part, then this characteristic's equation represents an unstable system.

Example 9 : *For a system with characteristic equation* $F(s) = s^6 + 3s^5 + 4s^4 + 6s^3 + 5s^2 + 3s + 2 = 0$, *examine stability.*

Solution :

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	0	0	0	0

Row of zeros

$$P(s) = 2s^4 + 4s^2 + 2 = 0 \quad \text{i.e.} \quad s^4 + 2s^2 + 1 = 0$$

$$\frac{dP(s)}{ds} = 4s^3 + 4s$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	0	0	0	0

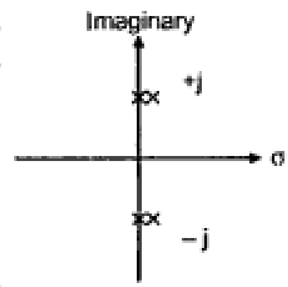
Row of zeros again

$$\therefore \qquad P'(s) = 2s^2 + 2 = 0$$

$$\frac{dP'(s)}{ds} = 4s = 0$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	4	0	0	0
s^0	2	0	0	0

No sign change, hence no root is located in R.H.S. of s-plane. As row of zeros occur, system may be marginally stable or unstable. To examine that find the roots of first auxiliary equation.



$P(s) = s^4 + 2s^2 + 1 = 0 \qquad s^2 = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

$s^2 = -1, \qquad s^2 = -1, \qquad s_{1,2} = \pm j, \qquad s_{3,4} = \pm j$

The roots of $P'(s) = 0$ are the roots of $P(s) = 0$. So do not solve second auxiliary equation. Predict the stability from the nature of roots of first auxiliary equation.

As there are repeated roots on imaginary axis, system is unstable.

Example 10 : *$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$. Check the stability of given characteristic equation using Routh's method.*

Solution :

s^6	1	8	20	16	
s^5	2	12	16	0	
s^4	2	12	16	0	
s^3	0	0	0	0	⟸ Special case 2

Row of zeros

$$P(s) = 2s^4 + 12s^2 + 16 = 0$$

$$\frac{dP}{ds} = 8s^3 + 24s = 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	
s^1	2.67	0		
s^0	16			

No sign change, so system may be stable. But as there is row of zero, system will be (i) marginally stable or (ii) unstable. To examine this solve $P(s) = 0$.

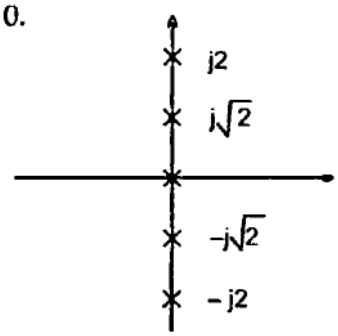
$$2s^4 + 12s^2 + 16 = 0$$

$$s^4 + 6s^2 + 8 = 0$$

Put $s^2 = y$

$$\therefore \quad y^2 + 6y + 8 = 0$$

$$y = -6 \pm \frac{\sqrt{36-32}}{2}$$



$$= -3 \pm 1 = -2, -4$$

$$\therefore \quad s^2 = -2 \quad \text{and} \quad s^2 = -4$$

$$\therefore \quad s = \pm j\sqrt{2} \quad \text{and} \quad s = \pm j2$$

Nonrepeated roots on imaginary axis. Hence system is marginally stable.